The equation may be written

$$\frac{1}{y_{x+1}} - \frac{1}{y_x} = \frac{y_{x+1}}{xy_x}.$$

Hence

$$\frac{1}{y_x} - \frac{1}{y_1} = \sum_1^{x-1} \frac{1}{x} \cdot \frac{y_{x+1}}{y_x} = \sum_1^{x-1} \frac{1}{x}\left[1 - \frac{y_x - y_{x+1}}{y_x}\right] = \sum_1^{x-1} \frac{1}{x} - \sum_1^{x-1} \frac{1}{x^2} \cdot \frac{y^2{}_{x+1}}{y_x}.$$

Now

$$\sum_1^{x-1} \frac{1}{x} = \log x + r,$$

where r approaches Euler's constant C when x becomes infinite (Goursat-Hedrick, vol. 1, p. 103, Ex. 1); so that, if we substitute and divide by $\log x$, we have

$$\frac{1}{y_x \log x} - 1 = \frac{\dfrac{1}{y_1} + r - \displaystyle\sum_1^{x-1} \frac{1}{x^2} \cdot \frac{y^2{}_{x+1}}{y_x}}{\log x}.$$

But from (2) $y_{x+1} < y_x < y_1$, $y^2{}_{x+1} < y_1 y_x$. Therefore

$$\sum_1^{x-1} \frac{1}{x^2} \cdot \frac{y^2{}_{x+1}}{y_x} < y_1 \sum_1^{x-1} \frac{1}{x^2} < \frac{\pi^2 y_1}{6}.$$

The numerator of the fraction remains numerically less than a fixed number when x becomes infinite and we have

$$\lim y_x \log x = 1.$$

It is to be noticed that the above method can be immediately extended to

$$y_x = f(x) y^2{}_{x+1} + y_{x+1},$$

for certain functions $f(x)$, where $\sum_1^{\infty} [f(x)]^2$ is a convergent series.

NOTE. From (1) we obtain

$$\frac{y_x}{y_{x+1}} - 1 = \frac{y_{x+1}}{x}, \qquad \text{and} \qquad x\left(\frac{1}{y_{x+1}} - \frac{1}{y_x}\right) = \frac{y_{x+1}}{y_x}. \tag{3}$$

When x becomes infinite, the first equation in (3) together with (2) shows that y_x/y_{x+1} approaches unity. Hence the second equation gives

$$\lim_{x \to \infty} x\left(\frac{1}{y_{x+1}} - \frac{1}{y_x}\right) = 1.$$

It is easily seen that the limit is the same if we replace x by $x + 1$ and when this is done it follows that

$$\lim_{x \to \infty} \frac{1}{\log (x+1) y_{x+1}} = 1,$$

by use of the theorem on page 108, § 162, E. Cesàro, *Elementares Lehrbuch der algebraischen Analysis . . .*, Leipzig, 1904. The desired result easily follows from the above.

2846 [1920, 326].

Find the entire volume within the surface $x^{1/2} + y^{1/2} + z^{1/2} = a^{1/2}$. (W. A. Granville, *Elements of Differential and Integral Calculus*, revised ed., 1911, p. 420.)

This equation, rationalized, is the equation of Steiner's quartic surface, every tangent plane to which cuts it in two conics. (Cf. Salmon-Rogers, *Analytic Geometry of Three Dimensions*, 5th ed., vol. 2, 1915, pp. 171, 201, 207, 213f. Also C. M. Jessop, *Quartic Surfaces* 1916, chapter 7.)

I. SOLUTION BY L. A. EASTBURN, North Arizona Normal School, Flagstaff, Ariz.

The required volume inclosed by the surface is

$$v = \int_0^a \int_0^{(a^{1/2} - x^{1/2})^2} \int_0^{(a^{1/2} - x^{1/2} - y^{1/2})^2} dz dy dx,$$

$$= \int_0^a \int_0^{(a^{1/2}-x^{1/2})^2} [(a^{1/2} - x^{1/2})^2 - 2(a^{1/2} - x^{1/2})y^{1/2} + y]dydx,$$

$$= 1/6 \int_0^a [a^2 - 4a^{3/2}x^{1/2} + 6ax - 4a^{1/2}x^{3/2} + x^2]dx,$$

$$= 1/6 \left[a^3 - 8/3a^3 + 3a^3 - 8/5a^3 + \frac{a^3}{3} \right] = \frac{a^3}{90}.$$

II. Notes by R. C. Archibald, Brown University.

The integral here arising is a particular case of one considered by Lejeune Dirichlet in 1839.[1] If $V = \int x^{a-1}y^{b-1}z^{c-1} \cdots dxdydz \cdots$, considering all positive values of $x, y, z, \cdots$ such that

$$\left(\frac{x}{\alpha}\right)^p + \left(\frac{y}{\beta}\right)^q + \left(\frac{z}{\gamma}\right)^r + \cdots < 1$$

the constants $a, b, c, \cdots, p, q, r, \cdots, \alpha, \beta, \gamma, \cdots$ being also positive, then

$$V = \frac{\alpha^a\beta^b\gamma^c \cdots}{pqr \cdots} \frac{\Gamma\left(\frac{a}{p}\right)\Gamma\left(\frac{b}{q}\right)\Gamma\left(\frac{c}{r}\right)\cdots}{\Gamma\left(1 + \frac{a}{p} + \frac{b}{q} + \frac{c}{r} + \cdots\right)}.$$

For the surface $(x/a)^{1/n} + (y/b)^{1/n} + (z/c)^{1/n} = 1$, the volume in the first octant would be $abc(n!)^3/(3n)!$ When $n = 7/2$ we have a result given[2] in 1883

$$V = \frac{abc}{(2/7)^3} \cdot \frac{[\Gamma(7/2)]^3}{\Gamma(23/2)}.$$

The values, for $n = 3/2$ and $n = 1/2$ were given in Williamson, *Elementary Treatise on the Integral Calculus*, 6 ed., 1891, pp. 289–290. The result for the surface $(x/a)^2 + (y/b)^2 + (z/c)^4 = 1$, given in Todhunter, *A Treatise on the Integral Calculus*, 4 ed., 1874, p. 186, follows at once from the Dirichlet formula above.

2852 [1920, 377]. Proposed by D. H. RICHERT, Bethel College, Newton, Kans.

What is the radius of a cylinder inscribed in a right cone, radius of base $R = 5$ inches, and altitude $h = 18$ inches, the volume of the cylinder to be $1/n$ $(= 3/4)$ that of the cone?

Solution by H. S. Uhler, Yale University.

Let V denote the volume of the cone, and let r, v, and z denote, respectively, the radius, the volume, and the altitude of the cylinder. Then

$$V = \tfrac{1}{3}\pi hR^2, \qquad v = \pi zr^2, \qquad \text{and} \qquad v = \frac{1}{n}V;$$

hence,

$$zr^2 = \frac{hR^2}{3n}. \tag{1}$$

From the similar right triangles obtained by passing a plane through the common axis of the cone and cylinder we find

$$z = \frac{h(R - r)}{R}. \tag{2}$$

These two equations are homogeneous in h and z, and also in R and r; therefore they determine the ratio of the altitudes and the ratio of the radii independently as functions of n alone.

Substituting from (2) for z in equation (1) we obtain

$$r^3 - Rr^2 + R^3/(3n) = 0.$$

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[1] *Comptes Rendus . . . de l'Académie des Sciences*, vol. 8, p. 156; also in *Journal de Mathématiques Pures et Appliquées*, vol. 4, p. 168.

[2] *Mathematical Questions with their solutions from the "Educational Times,"* vol. 38, p. 104.